Existence of positive solutions to quasi-linear elliptic equations with exponential growth in the whole Euclidean space

Yunyan Yang

Department of Mathematics, Renmin University of China, Beijing 100872, P. R. China

Abstract

In this paper a quasi-linear elliptic equation in the whole Euclidean space is considered. The nonlinearity of the equation is assumed to have exponential growth or have critical growth in view of Trudinger-Moser type inequality. Under some assumptions on the potential and the nonlinearity, it is proved that there is a nontrivial positive weak solution to this equation. Also it is shown that there are two distinct positive weak solutions to a perturbation of the equation. The method of proving these results is combining Trudinger-Moser type inequality, Mountain-pass theorem and Ekeland's variational principle.

Key words: Trudinger-Moser inequality, singular Trudinger-Moser inequality, N-Laplace equation, exponential growth
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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. There are fruitful results on the following problem

$$\begin{cases} -\Delta_p u = f(x,u) \quad \text{in} \quad \Omega \\ u \in W_0^{1,p}(\Omega), \end{cases} \tag{1.1}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$. When $p = 2$ and $|f(x,u)| \le c(|u|+|u|^{q-1})$, $1 < q \le 2^* = 2N/(N-2)$, $N \ge 3$. Among pioneer works we mention Brézis [8], Brézis-Nirenberg [10], Bartsh-Willem [11] and Capozzi-Fortunato-Palmieri [13]. For $p \le N$ and $p^2 \le N$, Garcia-Alonso [23] generalized Brézis-Nirenberg's existence and nonexistence results to p-Laplace equation. When $\Omega = \mathbb{R}^N$ and $p = 2$, one may consider the semilinear Schrödinger equation instead of (1.1):

$$\begin{cases} -\Delta u + V(x)u = f(x,u) \quad \text{in} \quad \mathbb{R}^N \\ u \in W^{1,N}(\mathbb{R}^N), \end{cases} \tag{1.2}$$

where again $|f(x,u)| \le c(|u| + |u|^{q-1})$, $1 < q \le 2^* = 2N/(N-2)$. Many papers are devoted to (1.2), we refer the reader to Kryszewski-Szulkin [25], Alama-Li [6], Ding-Ni [17] and Jeanjean [24]. Sobolev embedding theorem and the critical point theory, particularly moumtain-pass

Email address: yunyanyang@ruc.edu.cn (Yunyan Yang)
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theorem would play an important role in studying problems (1.1) and (1.2) since both of them have variational structure. When $p = N$ and $f(x,u)$ behaves like $e^{\alpha|u|^{N/(N-1)}}$ as $|u| \to \infty$, problem (1.1) was studied by Adimurthi [2], Adimurthi-Yadava[4], Ruf et al [15, 16], J. M. do Ó [19], Panda [30] and the references therein. To the author's knowledge, all theses results are based on Trudinger-Moser inequality [28, 31, 34] and critical point theory.

In this paper we consider the existence of positive solutions of the quasi-linear equation

$$-\Delta_N u + V(x)|u|^{N-2}u = \frac{f(x,u)}{|x|^\beta}, \quad x \in \mathbb{R}^N \ (N \geq 2), \tag{1.3}$$

where $\Delta_N u = \operatorname{div}(|\nabla u|^{N-2}\nabla u)$, $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function, $f(x,s)$ is continuous in $\mathbb{R}^N \times \mathbb{R}$ and behaves like $e^{\alpha s^{N/(N-1)}}$ as $s \to +\infty$, and $0 \leq \beta < N$. Problem (1.3) can be compared with (1.2) in this way: Sobolev embedding theorem can be applied to (1.2), while Trudinger-Moser type embedding theorem can be applied to (1.3). When $\beta = 0$, problem (1.3) was studied by D. Cao [12] in the case $N = 2$, by Panda [29], J. M. do Ó [18] and Alves-Figueiredo [7] in general dimensional case. When $0 < \beta < N$, problem (1.3) is closely related to a singular Trudinger-Moser type inequality, namely

Theorem A ([5]). *For all $\alpha > 0$, $0 \leq \beta < N$, and $u \in W^{1,N}(\mathbb{R}^N)$ $(N \geq 2)$, there holds*

$$\int_{\mathbb{R}^N} \frac{e^{\alpha|u|^{N/(N-1)}} - \sum_{k=0}^{N-2} \frac{\alpha^k |u|^{kN/(N-1)}}{k!}}{|x|^\beta} dx < \infty. \tag{1.4}$$

Furthermore, we have for all $\alpha \leq \left(1 - \frac{\beta}{N}\right)\alpha_N$ and $\tau > 0$,

$$\sup_{\int_{\mathbb{R}^N}(|\nabla u|^N + \tau|u|^N)dx \leq 1} \int_{\mathbb{R}^N} \frac{e^{\alpha|u|^{N/(N-1)}} - \sum_{k=0}^{N-2} \frac{\alpha^k |u|^{kN/(N-1)}}{k!}}{|x|^\beta} dx < \infty. \tag{1.5}$$

This inequality is sharp : for any $\alpha > \left(1 - \frac{\beta}{N}\right)\alpha_N$, the supremum is infinity.

This theorem extends a result of Adimurthi-Sandeep [3] on a bounded smooth domain. When $\beta = 0$ and $\tau = 1$, (1.5) was proved by B. Ruf in the case $N = 2$ via symmetrization method and by Li-Ruf [26] in general dimensional case via the method of blow-up analysis. When $\beta = 0$ and $\alpha < \alpha_N$, (1.5) was first proved by Cao [12] in the case $N = 2$, and then by Panda [29], J. M. do Ó [18] in general dimensional case. A similar but different type inequality was obtained by Adachi-Tanaka [1].

We assume the following two conditions on the potential $V(x)$:

(V_1) $V(x) \geq V_0 > 0$ in $\mathbb{R}^N$ for some $V_0 > 0$;

(V_2) The function $\frac{1}{V(x)}$ belongs to $L^{\frac{1}{N-1}}(\mathbb{R}^N)$.

As for the nonlinearity $f(x,s)$ we suppose the following:

(H_1) There exist constants $\alpha_0, b_1, b_2 > 0$ such that for all $(x,s) \in \mathbb{R}^N \times \mathbb{R}^+$,

$$|f(x,s)| \leq b_1 s^{N-1} + b_2 \left\{e^{\alpha_0|s|^{N/(N-1)}} - S_{N-2}(\alpha_0, s)\right\};$$

(H_2) There exists $\mu > N$ such that for all $x \in \mathbb{R}^N$ and $s > 0$,

$$0 < \mu F(x, s) \equiv \mu \int_0^s f(x, t) dt \leq s f(x, s);$$

(H_3) There exist constant R_0, $M_0 > 0$ such that for all $x \in \mathbb{R}^N$ and $s \geq R_0$,

$$F(x, s) \leq M_0 f(x, s).$$

Define a function space

$$E = \left\{ u \in W^{1,N}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^N dx < \infty \right\}. \tag{1.6}$$

We say that $u \in E$ is a weak solution of problem (1.3) if for all $\varphi \in E$ we have

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{N-2} \nabla u \nabla \varphi + V(x)|u|^{N-2} u \varphi \right) dx = \int_{\mathbb{R}^N} \frac{f(x, u)}{|x|^\beta} \varphi dx.$$

The assumption (V_1) implies that E is a reflexive Banach space when equipped with the norm

$$\|u\|_E \equiv \left\{ \int_{\mathbb{R}^N} \left(|\nabla u|^N + V(x)|u|^N \right) dx \right\}^{\frac{1}{N}} \tag{1.7}$$

and for any $q \geq N$, the embedding

$$E \hookrightarrow W^{1,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$$

is continuous. However (V_2) together with (V_1) implies that $E \hookrightarrow L^q(\mathbb{R}^N)$ is compact for all $q \geq 1$ (see Lemma 2.4 below). Surprisingly the assumption (V_2) is much better than

(V_2') $V(x) \to +\infty$ as $|x| \to +\infty$,

since (V_1) together with (V_2') only leads to the compact embedding $E \hookrightarrow L^q(\mathbb{R}^N)$ for all $q \geq N$ (see for example Costa [14] for details). This is the case in [5, 21]. However in this paper our argument of proving main results seriously depends on the compact embedding $E \hookrightarrow L^q(\mathbb{R}^N)$ for all $q \geq 1$.

For any $\beta : 0 \leq \beta < N$, we define a singular eigenvalue for the N-Laplace operator by

$$\lambda_\beta = \inf_{u \in E,\, u \not\equiv 0} \frac{\|u\|_E^N}{\int_{\mathbb{R}^N} \frac{|u|^N}{|x|^\beta} dx}. \tag{1.8}$$

It is easy to see that $\lambda_\beta > 0$. Write $m(r) = \sup_{|x| \leq r} V(x)$ and

$$\mathcal{M} = \inf_{r > 0} \frac{(N - \beta)^N}{\alpha_0^{N-1} r^{N-\beta}} e^{(N-\beta) m(r) \frac{(N-2)!}{N^N} r^N}, \tag{1.9}$$

where α_0 is given by (H_1). If $V(x)$ is continuous and (V_1) is satisfied, then $m(r)$ is a positive continuous function and $\mathcal{M}$ can be attained by some $r > 0$.

One of our main results can be stated as follows:

Theorem 1.1. *Assume that $V(x)$ is a continuous function satisfying (V_1) and (V_2). $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function and the hypothesis (H_1), (H_2) and (H_3) hold. Furthermore we assume*

(H_4) $\qquad \limsup_{s\to 0+} \frac{N|F(x,s)|}{s^N} < \lambda_\beta$ uniformly with respect to $x \in \mathbb{R}^N$;

(H_5) $\qquad \liminf_{s\to +\infty} s f(x,s) e^{-\alpha_0 s^{\frac{N}{N-1}}} = \beta_0 > \mathcal{M}$ uniformly with respect to $x \in \mathbb{R}^N$.

Then the equation (1.3) has a nontrivial positive mountain-pass type weak solution.

Here and throughout this paper, we say that a weak solution u is positive if $u(x) \geq 0$ for almost every $x \in \mathbb{R}^N$. It should be pointed out that $\mathcal{M}$ is not the best constant in (H_5). It would be interesting if one can find an explicit smaller number replacing $\mathcal{M}$.

In [5], Theorem A has been employed to study a perturbation of the equation (1.3), namely

$$-\Delta_N u + V(x)|u|^{N-2}u = \frac{f(x,u)}{|x|^\beta} + \epsilon h, \quad x \in \mathbb{R}^N \ (N \geq 2), \tag{1.10}$$

where $\epsilon > 0$ is a constant and $h : \mathbb{R}^N \to \mathbb{R}$ is a function belonging to E^*, the dual space of E. If $V(x)$ satisfies (V_1), (V_2'), and $f(x,s)$ satisfies $(H_1) - (H_4)$, then it was shown in [5] that when $\epsilon > 0$ is sufficiently small and $h \not\equiv 0$, the problem (1.10) has two weak solutions: one is of mountain-pass type and the other is of negative energy. But we can not conclude that the two solutions are distinct. In this paper, replacing (V_2') by (V_2) and imposing additional condition (H_5), we can prove that the above two solutions are distinct, namely

Theorem 1.2. *Suppose that $f(x,s)$ is continuous in $\mathbb{R}^N \times \mathbb{R}$ and $(H_1) - (H_5)$ hold. $V(x)$ is continuous in $\mathbb{R}^N$ satisfying (V_1) and (V_2), h belongs to E^*, the dual space of E, with $h \geq 0$ and $h \not\equiv 0$. Then there exists $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$, then the problem (1.10) has two distinct positive weak solutions.*

The proof of Theorem 1.1 and Theorem 1.2 is based on Theorem A, the mountain-pass theorem without the Palais-Smale condition [32] and the Ekeland's variational principle [35], which were also used in [5, 21]. Let us make some reduction on problems (1.3) and (1.10). Set

$$\widetilde{f}(x,s) = \begin{cases} 0, & f(x,s) < 0 \\ f(x,s), & f(x,s) \geq 0. \end{cases}$$

Assume $u \in E$ is a weak solution of

$$-\Delta_N u + V(x)|u|^{N-2}u = \frac{\widetilde{f}(x,u)}{|x|^\beta} + \epsilon h, \tag{1.11}$$

where $h \geq 0$ and $\epsilon > 0$, then the negative part of u, namely

$$u_-(x) = \begin{cases} 0, & u(x) > 0 \\ u(x), & u(x) \leq 0 \end{cases}$$

belongs to the function space E and satisfies

$$\begin{aligned}\int_{\mathbb{R}^N}(|\nabla u_-|^N + V(x)|u_-|^N)dx &= \int_{\mathbb{R}^N}\frac{\widetilde{f}(x,u)}{|x|^\beta}u_-dx + \epsilon\int_{\mathbb{R}^N}hu_-dx\\ &= \epsilon\int_{\mathbb{R}^N}hu_-dx \leq 0.\end{aligned}$$

Hence $u_-(x) = 0$ for almost every $x \in \mathbb{R}^N$ and thus u is a positive weak solution of (1.11). This together with (H_2) implies $f(x,u) \geq 0$. It follows that $\widetilde{f}(x,u) = f(x,u)$. Therefore u is also a positive weak solution of (1.10). When $h = 0$, (1.10) becomes (1.3). Based on this, to prove Theorems 1.1 and 1.2, it suffices to find weak solutions of (1.3) and (1.10) with f replaced by $\widetilde{f}$ respectively. So throughout this paper, we can assume without loss of generality

$$f(x,s) \equiv 0, \quad \forall s < 0. \tag{1.12}$$

Before ending this introduction, we would like to mention that results similar to Theorem 1.2 in two dimensional case, i.e. $N = 2$, was obtained by J. M. do Ó [21]. Similar problems for bi-Laplace equation in $\mathbb{R}^4$ was considered by the author in [36]. For compact Riemannian manifold case, we refer the reader to [22, 37]. Also it should be remarked that results obtained in [5] and in the present paper still hold if there is only the subcritical case of (1.5), namely for any $\alpha < (1-\beta/N)\alpha_N$ and $\tau > 0$,

$$\sup_{\int_{\mathbb{R}^N}(|\nabla u|^N+\tau|u|^N)dx\leq 1}\int_{\mathbb{R}^N}\frac{e^{\alpha|u|^{N/(N-1)}} - \sum_{k=0}^{N-2}\frac{\alpha^k|u|^{kN/(N-1)}}{k!}}{|x|^\beta}dx < \infty.$$

In fact, in [7, 12, 18, 29], all the contributors only used the above subcritical inequality.

The remaining part of this paper is organized as follows: In section 2, we display several key estimates in later compactness analysis. In section 3, we consider the functionals related to problems (1.3) and (1.10). Finally Theorem 1.1 is proved in section 4 and Theorem 1.2 is proved in section 5.

2. Key estimates

In this section we will derive several technical lemmas for our use later. For any integer $N \geq 2$ and real number s, we define a function $\zeta : \mathbb{N}\times\mathbb{R} \to \mathbb{R}$ by

$$\zeta(N,s) = e^s - \sum_{k=0}^{N-2}\frac{s^k}{k!} = \sum_{k=N-1}^{\infty}\frac{s^k}{k!}. \tag{2.1}$$

Lemma 2.1. *Let $s \geq 0$, $p \geq 1$ be real numbers and $N \geq 2$ be an integer. Then there holds*

$$(\zeta(N,s))^p \leq \zeta(N,ps). \tag{2.2}$$

Proof. We prove (2.2) by induction with respect to N. Define a function

$$\phi(s) = (e^s-1)^p - (e^{ps}-1).$$

It is easy to see that for $s \geq 0$ and $p \geq 1$,

$$\phi'(s) = p(e^s - 1)^{p-1} - pe^{ps} \leq 0.$$

Hence $\phi(s) \leq \phi(0) = 0$ and thus (2.2) holds for $N = 2$. Suppose (2.2) holds for $N \geq 2$, we only need to prove that

$$(\zeta(N+1, s))^p \leq \zeta(N+1, ps). \tag{2.3}$$

For this purpose we set

$$\psi(s) = \left(e^s - \sum_{k=0}^{N-1} \frac{s^k}{k!}\right)^p - \left(e^{ps} - \sum_{k=0}^{N-1} \frac{(ps)^k}{k!}\right).$$

A straightforward calculation shows

$$\begin{aligned}
\psi'(s) &= p\left(e^s - \sum_{k=0}^{N-1} \frac{s^k}{k!}\right)^{p-1} \left(e^s - \sum_{k=1}^{N-1} \frac{s^{k-1}}{(k-1)!}\right) \\
&\quad - \left(pe^{ps} - p\sum_{k=1}^{N-1} \frac{(ps)^{k-1}}{(k-1)!}\right) \\
&\leq p\left\{\left(e^s - \sum_{k=1}^{N-1} \frac{s^{k-1}}{(k-1)!}\right)^p - \left(e^{ps} - \sum_{k=1}^{N-1} \frac{(ps)^{k-1}}{(k-1)!}\right)\right\} \\
&= p\left\{\left(e^s - \sum_{k=0}^{N-2} \frac{s^k}{k!}\right)^p - \left(e^{ps} - \sum_{k=0}^{N-2} \frac{(ps)^k}{k!}\right)\right\} \leq 0.
\end{aligned}$$

Here we have used the induction assumption $(\zeta(N, s))^p \leq \zeta(N, ps)$. Thus $\psi(s) \leq \psi(0) = 0$ for $s \geq 0$, and whence (2.3) holds. Therefore (2.2) holds for any integer $N \geq 2$. □

Lemma 2.2. *For all $N \geq 2$, $s \geq 0$, $t \geq 0$, $\mu > 1$ and $\nu > 1$ with $1/\mu + 1/\nu = 1$, there holds*

$$\zeta(N, s+t) \leq \frac{1}{\mu}\zeta(N, \mu s) + \frac{1}{\nu}\zeta(N, \nu t).$$

Proof. Observing that

$$\frac{\partial^2}{\partial s^2}\zeta(2, s) = e^s \geq 0, \quad \frac{\partial^2}{\partial s^2}\zeta(3, s) = e^s \geq 0$$

and when $N \geq 4$,

$$\frac{\partial^2}{\partial s^2}\zeta(N, s) = e^s - \sum_{k=2}^{N-2} \frac{s^{k-2}}{(k-2)!} = e^s - \sum_{k=0}^{N-4} \frac{s^k}{k!} \geq 0,$$

we conclude that $\zeta(N, s)$ is convex with respect to s for all $N \geq 2$. Hence

$$\zeta(N, s+t) = \zeta\left(N, \frac{1}{\mu}\mu s + \frac{1}{\nu}\nu t\right) \leq \frac{1}{\mu}\zeta(N, \mu s) + \frac{1}{\nu}\zeta(N, \nu t).$$

This concludes the lemma. □

Lemma 2.3. *Let (w_n) be a sequence in E. Suppose $\|w_n\|_E = 1$, $w_n \rightharpoonup w_0$ weakly in E, $w_n(x) \to w_0(x)$ and $\nabla w_n(x) \to \nabla w_0(x)$ for almost every $x \in \mathbb{R}^N$. Then for any $p : 0 < p < \frac{1}{(1-\|w_0\|_E^N)^{1/(N-1)}}$*

$$\sup_n \int_{\mathbb{R}^N} \frac{\zeta\left(N, \alpha_N(1-\beta/N)p|w_n|^{\frac{N}{N-1}}\right)}{|x|^\beta} dx < \infty. \tag{2.4}$$

Proof. Noticing that

$$|w_n|^{\frac{N}{N-1}} = |w_n - w_0 + w_0|^{\frac{N}{N-1}} \le (1+\epsilon)|w_n - w_0|^{\frac{N}{N-1}} + c(\epsilon)|w_0|^{\frac{N}{N-1}},$$

we have by using Lemma 2.2

$$\begin{aligned}
\zeta\left(N, \alpha_N(1-\beta/N)p|w_n|^{\frac{N}{N-1}}\right) &\le \frac{1}{\mu}\zeta\left(N, \mu(1+\epsilon)\alpha_N(1-\beta/N)p|w_n - w_0|^{\frac{N}{N-1}}\right) \\
&\quad + \frac{1}{\nu}\zeta\left(N, \nu c(\epsilon)\alpha_N(1-\beta/N)p|w_0|^{\frac{N}{N-1}}\right) \\
&\le \zeta\left(N, \mu(1+\epsilon)\alpha_N(1-\beta/N)p\|w_n - w_0\|_E^{\frac{N}{N-1}}\left(\frac{|w_n - w_0|}{\|w_n - w_0\|_E}\right)^{\frac{N}{N-1}}\right) \\
&\quad + \zeta\left(N, \nu c(\epsilon)\alpha_N(1-\beta/N)p|w_0|^{\frac{N}{N-1}}\right),
\end{aligned}$$

where $\mu > 1$, $\nu > 1$ and $1/\mu + 1/\nu = 1$. By Brézis-Lieb's Lemma [9],

$$\|w_n - w_0\|_E^N = 1 - \|w_0\|_E^N + o_n(1),$$

where $o_n(1) \to 0$ as $n \to \infty$. Hence for any $p : 0 < p < \frac{1}{(1-\|w_0\|_E^N)^{1/(N-1)}}$, one can choose $\epsilon > 0$ sufficiently small and $\mu > 1$ sufficiently close to 1 such that

$$\mu(1+\epsilon)\alpha_N(1-\beta/N)p\|w_n - w_0\|_E^{\frac{N}{N-1}} < \alpha_N(1-\beta/N).$$

Now (2.4) follows from Theorem A immediately. □

Lemma 2.4. *Assume $V : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is continuous and (V_1), (V_2) hold. Then E is compactly embedded in $L^q(\mathbb{R}^N)$ for all $q \ge 1$.*

Proof. By (V_1), the standard Sobolev embedding theorem implies that the following embedding is continuous

$$E \hookrightarrow W^{1,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) \quad \text{for all} \quad N \le q < \infty.$$

It follows from the Hölder inequality and (V_2) that

$$\int_{\mathbb{R}^N} |u| dx \le \left(\int_{\mathbb{R}^N} \frac{1}{V^{\frac{1}{N-1}}} dx\right)^{1-1/N} \left(\int_{\mathbb{R}^N} V|u|^N dx\right)^{1/N} \le \left(\int_{\mathbb{R}^N} \frac{1}{V^{\frac{1}{N-1}}} dx\right)^{1-1/N} \|u\|_E.$$

For any $\gamma : 1 < \gamma < N$, there holds

$$\int_{\mathbb{R}^N} |u|^\gamma dx \le \int_{\mathbb{R}^N} (|u| + |u|^N) dx \le \left(\int_{\mathbb{R}^N} \frac{1}{V^{\frac{1}{N-1}}} dx\right)^{1-1/N} \|u\|_E + \frac{1}{V_0}\|u\|_E^N,$$

where V_0 is given by (V_1). Thus we get continuous embedding $E \hookrightarrow L^q(\mathbb{R}^N)$ for all $q \geq 1$.

To prove that the above embedding is also compact, take a sequence of functions $(u_k) \subset E$ such that $\|u_k\|_E \leq C$ for all k, we must prove that up to a subsequence there exists some $u \in E$ such that u_k convergent to u strongly in $L^q(\mathbb{R}^N)$ for all $q \geq 1$. Without loss of generality we may assume

$$\begin{cases} u_k \rightharpoonup u & \text{weakly in} \quad E \\ u_k \to u & \text{strongly in} \quad L^q_{\text{loc}}(\mathbb{R}^N), \ \forall q \geq 1 \\ u_k \to u & \text{almost everywhere in} \quad \mathbb{R}^N. \end{cases} \tag{2.5}$$

In view of (V_2), for any $\epsilon > 0$, there exists $R > 0$ such that

$$\left(\int_{|x|>R} \frac{1}{V^{\frac{1}{N-1}}} dx \right)^{1-1/N} < \epsilon.$$

Hence

$$\int_{|x|>R} |u_k - u| dx \leq \left(\int_{\mathbb{R}^N} \frac{1}{V^{\frac{1}{N-1}}} dx \right)^{1-1/N} \left(\int_{\mathbb{R}^N} V |u|^N dx \right)^{1/N} \leq \epsilon \|u_k - u\|_E \leq C\epsilon. \tag{2.6}$$

Here and in the sequel we often denote various constants by the same C. On the other hand, it follows from (2.5) that $u_k \to u$ strongly in $L^1(\mathbb{B}_R(0))$, where $\mathbb{B}_R(0) \subset \mathbb{R}^N$ is the ball centered at 0 with radius R. This together with (2.6) leads to

$$\limsup_{k\to\infty} \int_{\mathbb{R}^N} |u_k - u| dx \leq C\epsilon.$$

Since ϵ is arbitrary, we obtain

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} |u_k - u| dx = 0.$$

For $q > 1$, it follows from the continuous embedding $E \hookrightarrow L^s(\mathbb{R}^N)$ $(s \geq 1)$ that

$$\begin{aligned} \int_{\mathbb{R}^N} |u_k - u|^q dx &= \int_{\mathbb{R}^N} |u_k - u|^{\frac{1}{2}} |u_k - u|^{(q-\frac{1}{2})} dx \\ &\leq \left(\int_{\mathbb{R}^N} |u_k - u| dx \right)^{1/2} \left(\int_{\mathbb{R}^N} |u_k - u|^{2q-1} dx \right)^{1/2} \\ &\leq C \left(\int_{\mathbb{R}^N} |u_k - u| dx \right)^{1/2} \to 0 \end{aligned}$$

as $k \to \infty$. This concludes the lemma. □

3. Functionals and compactness analysis

3.1. The functionals and their profiles

As we mentioned in the introduction, problems (1.3) and (1.10) have variational structure. To apply the critical point theory, we define the functional $J_{\beta,\epsilon} : E \to \mathbb{R}$ by

$$J_{\beta,\epsilon}(u) = \frac{1}{N} \|u\|_E^N - \int_{\mathbb{R}^N} \frac{F(x,u)}{|x|^\beta} dx - \epsilon \int_{\mathbb{R}^N} hu dx,$$

where $\epsilon \geq 0$, $0 \leq \beta < N$, $\|u\|_E$ is the norm of $u \in E$ defined by (1.7) and $F(x,s) = \int_0^s f(x,t)dt$ is the primitive of $f(x,s)$. Assume f satisfies the hypothesis (H_1). Then there exist some positive constants $\alpha_1 > \alpha_0$ and b_3 such that for all $(x,s) \in \mathbb{R}^N \times \mathbb{R}$, $F(x,s) \leq b_3\zeta(N, \alpha_1|s|^{N/(N-1)})$. Thus $J_{\beta,\epsilon}$ is well defined thanks to Theorem A. In the case $\epsilon = 0$, we denote $J_{\beta,0}$ for simplicity by

$$J(u) = \frac{1}{N}\|u\|_E^N - \int_{\mathbb{R}^N} \frac{F(x,u)}{|x|^\beta} dx.$$

The profiles of the functionals $J_{\beta,\epsilon}$ and $J(u)$ are well described in the following lemma.

Lemma 3.1. *Assume* (V_1), (H_1), (H_2), (H_3) *and* (H_4) *are satisfied. Then* (i) *for any nonnegative, compactly supported function* $u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}$, *there holds* $J_{\beta,\epsilon}(tu) \to -\infty$ *as* $t \to +\infty$; (ii) *there exists* $\epsilon_1 > 0$ *such that when* $0 < \epsilon < \epsilon_1$, *one can find* $r_\epsilon, \vartheta_\epsilon > 0$ *such that* $J_{\beta,\epsilon}(u) \geq \vartheta_\epsilon$ *for all* u *with* $\|u\|_E = r_\epsilon$, *where* r_ϵ *can be further chosen such that* $r_\epsilon \to 0$ *as* $\epsilon \to 0$. *When* $\epsilon = 0$, *there exists* $\delta > 0$ *and* $r > 0$ *such that* $J(u) \geq \delta$ *for all* $\|u\|_E = r$; (iii) *assume* $\epsilon > 0$ *and* $h \not\equiv 0$, *there exists a constant* $\tau > 0$ *such that if* $0 < t < \tau$, *then* $\inf_{\|u\|_E \leq t} J_{\beta,\epsilon}(u) < 0$.

Proof. We refer the reader to ([5], Lemmas 4.1, 4.2 and 4.3) for details. It is remarkable that we can also apply Lemma 2.1 and Lemma 2.4 instead of decreasing rearrangement argument in the proof of ([5], Lemma 4.2) and thus simplify it. □

To use the critical point theory, we need some regularity of the functionals $J_{\beta,\epsilon}$ and J. In fact, by Proposition 1 in [21] and standard arguments (see for example [32]), one can see that both $J_{\beta,\epsilon}$ and J belong to $C^1(E, \mathbb{R})$. A straightforward calculation shows

$$\langle J'(u), \phi \rangle = \int_{\mathbb{R}^N} \left(|\nabla u|^{N-2}\nabla u \nabla \phi + V|u|^{N-2}u\phi \right) dx - \int_{\mathbb{R}^N} \frac{f(x,u)}{|x|^\beta}\phi dx, \tag{3.1}$$

$$\langle J'_{\beta,\epsilon}(u), \phi \rangle = \int_{\mathbb{R}^N} \left(|\nabla u|^{N-2}\nabla u \nabla \phi + V|u|^{N-2}u\phi \right) dx - \int_{\mathbb{R}^N} \frac{f(x,u)}{|x|^\beta}\phi dx - \epsilon \int_{\mathbb{R}^N} h\phi dx \tag{3.2}$$

for all $\phi \in E$. Hence weak solutions of (1.3) and (1.10) are critical points of J and $J_{\beta,\epsilon}$ respectively.

3.2. *Min-Max value*

In this subsection, we prepare for estimating the min-max value of the functionals J and $J_{\beta,\epsilon}$. The idea is to construct a sequence of functions $M_n \in E$ and estimate $\max\limits_{t\geq 0} J(tM_n)$ and $\max\limits_{t\geq 0} J_{\beta,\epsilon}(tM_n)$. Recall Moser's function sequence

$$\widetilde{M}_n(x,r) = \frac{1}{\omega_{N-1}^{1/N}} \begin{cases} (\log n)^{1-1/N}, & |x| \leq r/n \\ \frac{\log \frac{r}{|x|}}{(\log n)^{1/N}}, & r/n < |x| \leq r \\ 0, & |x| > r. \end{cases}$$

Let $M_n(x,r) = \frac{1}{\|\widetilde{M}_n\|_E}\widetilde{M}_n(x,r)$. Then M_n belongs to E with its support in $\mathbb{B}_r(0)$ and $\|M_n\|_E = 1$.

Lemma 3.2. *Assume $V(x)$ is continuous and (V_1) is satisfied. Then there holds*

$$\|\widetilde{M}_n\|_E^N \le 1 + \frac{m(r)}{\log n}\left(\frac{(N-1)!}{N^N} r^N + o_n(1)\right), \tag{3.3}$$

where $m(r) = \max_{|x|\le r} V(x)$ and $o_n(1) \to 0$ as $n \to \infty$.

Proof. It is easy to calculate

$$\int_{\mathbb{R}^N} |\nabla \widetilde{M}_n|^N dx = \frac{1}{\omega_{N-1}} \int_{\frac{r}{n}\le|x|\le r} \frac{1}{|x|^N \log n} dx = 1.$$

Integration by parts gives

$$\begin{aligned}
\int_{\frac{r}{n}\le|x|\le r} \left(\log\frac{r}{|x|}\right)^N dx &= \omega_{N-1}\int_{\frac{r}{n}}^r s^{N-1}\left(\log\frac{r}{s}\right)^N ds \\
&= -\frac{\omega_{N-1}}{N}\left(\frac{r}{n}\right)^N (\log n)^N + \omega_{N-1}\int_{\frac{r}{n}}^r s^{N-1}\left(\log\frac{r}{s}\right)^{N-1} ds \\
&= -\omega_{N-1}\left(\frac{r}{n}\right)^N \left\{\frac{1}{N}(\log n)^N + \frac{1}{N}(\log n)^{N-1} + \frac{N-1}{N^2}(\log n)^{N-2}\right. \\
&\qquad\qquad \left. + \cdots + \frac{(N-1)(N-2)\cdots 3}{N^{N-2}}(\log n)^2\right\} \\
&\quad + \omega_{N-1}\frac{(N-1)!}{N^{N-2}}\int_{\frac{r}{n}}^r s^{N-1}\log\frac{r}{s} ds \\
&= \omega_{N-1}\frac{(N-1)!}{N^N} r^N + o_n(1).
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{\mathbb{R}^N} |\widetilde{M}_n|^N dx &= \frac{1}{\omega_{N-1}}\int_{|x|\le r/n} (\log n)^{N-1} dx + \frac{1}{\omega_{N-1}}\int_{\frac{r}{n}\le|x|\le r} \frac{\left(\log\frac{r}{|x|}\right)^N}{\log n} dx \\
&= \left(\frac{r}{n}\right)^N \frac{(\log n)^{N-1}}{N} + \frac{1}{\omega_{N-1}\log n}\int_{\frac{r}{n}\le|x|\le r}\left(\log\frac{r}{|x|}\right)^N dx \\
&= \frac{1}{\log n}\left(\frac{(N-1)!}{N^N} r^N + o_n(1)\right),
\end{aligned}$$

and thus

$$\begin{aligned}
\|\widetilde{M}_n\|_E^N &= \int_{\mathbb{R}^N} |\nabla \widetilde{M}_n|^N dx + \int_{\mathbb{R}^N} V(x)|\widetilde{M}_n|^N dx \\
&\le 1 + m(r)\int_{\mathbb{R}^N} |\widetilde{M}_n|^N dx \\
&= 1 + \frac{m(r)}{\log n}\left(\frac{(N-1)!}{N^N} r^N + o_n(1)\right).
\end{aligned}$$

This is exactly (3.3). □

Lemma 3.3. *Assume* (V_1), (H_1), (H_2), (H_3) *and* (H_5)*. There exists some* $n \in \mathbb{N}$ *such that*

$$\max_{t\geq 0} J(tM_n) < \frac{1}{N}\left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}. \tag{3.4}$$

Furthermore for the above n *there exists some* $\epsilon^* > 0$ *and* $\delta^* > 0$ *such that if* $0 \leq \epsilon < \epsilon^*$*, then*

$$\max_{t\geq 0} J_{\beta,\epsilon}(tM_n) < \frac{1}{N}\left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1} - \delta^*. \tag{3.5}$$

Proof. We first prove (3.4). By (H_5) and (1.9) (the definition of $\mathcal{M}$), there exists some $r > 0$ such that

$$\beta_0 > \frac{(N-\beta)^N}{\alpha_0^{N-1} r^{N-\beta}} e^{(N-\beta)m(r)\frac{(N-2)!}{N^N}r^N}. \tag{3.6}$$

Suppose by contradiction that for all $n \in \mathbb{N}$

$$\max_{t\geq 0} J(tM_n) \geq \frac{1}{N}\left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}. \tag{3.7}$$

By (i) of Lemma 3.1, $\forall n \in \mathbb{N}$, there exists $t_n > 0$ such that

$$J(t_n M_n) = \max_{t\geq 0} J(tM_n).$$

Thus (3.7) gives

$$J(t_n M_n) = \frac{t_n^N}{N} - \int_{\mathbb{R}^N} \frac{F(x, t_n M_n)}{|x|^\beta} dx \geq \frac{1}{N}\left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Noticing that $F(x, \cdot) \geq 0$, we have

$$t_n^N \geq \left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}. \tag{3.8}$$

It is easy to see that at $t = t_n$,

$$\frac{d}{dt}\left(\frac{t^N}{N} - \int_{\mathbb{R}^N} \frac{F(x, tM_n)}{|x|^\beta} dx\right) = 0,$$

or equivalently

$$t_n^N = \int_{\mathbb{R}^N} \frac{t_n M_n f(x, t_n M_n)}{|x|^\beta} dx. \tag{3.9}$$

By (H_5), $\forall \eta > 0$, $\exists R_\eta > 0$ such that for all $x \in \mathbb{R}^N$ and $u \geq R_\eta$

$$u f(x,u) \geq (\beta_0 - \eta) e^{\alpha_0 |u|^{\frac{N}{N-1}}}. \tag{3.10}$$

By Lemma 3.2, when $|x| \leq \frac{r}{n}$, we have

$$\begin{aligned} M_n^{\frac{N}{N-1}}(x,r) &\geq \frac{1}{\omega_{N-1}^{\frac{1}{N-1}}} \frac{\log n}{1 + \frac{1}{N-1}\frac{m(r)}{\log n}\left(\frac{(N-1)!}{N^N} r^N + o_n(1)\right)} \\ &= \omega_{N-1}^{-\frac{1}{N-1}} \log n - \omega_{N-1}^{-\frac{1}{N-1}} m(r) \frac{(N-2)!}{N^N} r^N + o_n(1). \end{aligned} \tag{3.11}$$

Hence we have by combining (3.9) and (3.10) that

$$
\begin{aligned}
t_n^N &\geq (\beta_0 - \eta) \int_{|x| \leq \frac{r}{n}} \frac{e^{\alpha_0 |t_n M_n|^{\frac{N}{N-1}}}}{|x|^\beta} dx \\
&= (\beta_0 - \eta) \int_{|x| \leq \frac{r}{n}} \frac{e^{\alpha_0 \omega_{N-1}^{-\frac{1}{N-1}} t_n^{\frac{N}{N-1}} \left(\log n - m(r) \frac{(N-2)!}{N^N} r^N + o_n(1)\right)}}{|x|^\beta} dx \\
&= (\beta_0 - \eta) \frac{\omega_{N-1}}{N-\beta} \left(\frac{r}{n}\right)^{N-\beta} e^{\alpha_0 \omega_{N-1}^{-\frac{1}{N-1}} t_n^{\frac{N}{N-1}} \left(\log n - m(r) \frac{(N-2)!}{N^N} r^N + o_n(1)\right)}. \qquad (3.12)
\end{aligned}
$$

This yields that t_n is a bounded sequence. In view of (3.8), we can also see from (3.12) that

$$
\lim_{n \to \infty} t_n^N = \left(\frac{N-\beta}{N} \frac{\alpha_N}{\alpha_0}\right)^{N-1}. \qquad (3.13)
$$

For otherwise there exists some $\delta > 0$ such that for sufficiently large n

$$
t_n^N \geq \left(\delta + \frac{N-\beta}{N} \frac{\alpha_N}{\alpha_0}\right)^{N-1}.
$$

Thus

$$
\alpha_0 \omega_{N-1}^{-\frac{1}{N-1}} t_n^{\frac{N}{N-1}} \geq N - \beta + \alpha_0 \omega_{N-1}^{-\frac{1}{N-1}} \delta
$$

and whence the right hand of (3.12) tends to infinity which contradicts the bounded-ness of t_n.

Now we estimate β_0. It follows from (3.9) and (3.10) that

$$
\begin{aligned}
t_n^N \geq\ & (\beta_0 - \eta) \int_{|x| \leq r} \frac{e^{\alpha_0 |t_n M_n|^{\frac{N}{N-1}}}}{|x|^\beta} dx + \int_{t_n M_n < R_\eta} \frac{t_n M_n f(x, t_n M_n)}{|x|^\beta} dx \\
& - (\beta_0 - \eta) \int_{t_n M_n < R_\eta} \frac{e^{\alpha_0 |t_n M_n|^{\frac{N}{N-1}}}}{|x|^\beta} dx. \qquad (3.14)
\end{aligned}
$$

Since $M_n \to 0$ almost everywhere in $\mathbb{R}^N$, we have by using the Lebesgue's dominated convergence theorem

$$
\lim_{n \to \infty} \int_{t_n M_n < R_\eta} \frac{t_n M_n f(x, t_n M_n)}{|x|^\beta} dx = 0, \qquad (3.15)
$$

$$
\lim_{n \to \infty} \int_{t_n M_n < R_\eta} \frac{e^{\alpha_0 |t_n M_n|^{\frac{N}{N-1}}}}{|x|^\beta} dx = \int_{|x| \leq r} \frac{1}{|x|^\beta} dx = \frac{\omega_{N-1} r^{N-\beta}}{N-\beta}. \qquad (3.16)
$$

Using (3.8),

$$
\int_{|x| \leq r} \frac{e^{\alpha_0 |t_n M_n|^{\frac{N}{N-1}}}}{|x|^\beta} dx \geq \int_{|x| \leq \frac{r}{n}} \frac{e^{\alpha_N (1-\beta/N) M_n^{\frac{N}{N-1}}}}{|x|^\beta} dx + \int_{\frac{r}{n} \leq |x| \leq r} \frac{e^{\alpha_N (1-\beta/N) M_n^{\frac{N}{N-1}}}}{|x|^\beta} dx. \qquad (3.17)
$$

On one hand we have by (3.11)

$$
\begin{aligned}
\int_{|x|\le\frac{r}{n}} \frac{e^{\alpha_N(1-\beta/N)M_n^{\frac{N}{N-1}}}}{|x|^\beta}dx &\ge e^{\alpha_N(1-\beta/N)\omega_{N-1}^{-\frac{1}{N-1}}\log n-\omega_{N-1}^{-\frac{1}{N-1}}m(r)\frac{(N-2)!}{N^N}r^N+o_n(1)}\int_{|x|\le\frac{r}{n}}\frac{1}{|x|^\beta}dx\\
&= \frac{\omega_{N-1}}{N-\beta}\left(\frac{r}{n}\right)^{N-\beta}e^{(N-\beta)\log n-(N-\beta)m(r)\frac{(N-2)!}{N^N}r^N+o_n(1)}\\
&= \frac{\omega_{N-1}r^{N-\beta}}{N-\beta}e^{-(N-\beta)m(r)\frac{(N-2)!}{N^N}r^N+o_n(1)}.
\end{aligned}
$$

On the other hand, by definition of M_n,

$$
\begin{aligned}
\int_{\frac{r}{n}\le|x|\le r} \frac{e^{\alpha_N(1-\beta/N)M_n^{\frac{N}{N-1}}}}{|x|^\beta}dx &= \int_{\frac{r}{n}\le|x|\le r}\frac{e^{(N-\beta)\left((\log n)^{-1/N}\|\widetilde{M}_n\|_E^{-1}\log\frac{r}{|x|}\right)^{\frac{N}{N-1}}}}{|x|^\beta}dx\\
&= \omega_{N-1}\int_{\frac{r}{n}}^{r} t^{N-\beta-1}e^{(N-\beta)\left((\log n)^{-1/N}\|\widetilde{M}_n\|_E^{-1}\log\frac{r}{t}\right)^{\frac{N}{N-1}}}dt\\
&= \omega_{N-1}r^{N-\beta}\int_0^{(\log n)^{1-1/N}\|\widetilde{M}_n\|_E^{-1}}(\log n)^{1/N}\|\widetilde{M}_n\|_E\\
&\qquad e^{(N-\beta)s^{\frac{N}{N-1}}-(N-\beta)\|\widetilde{M}_n\|_E(\log n)^{1/N}s}ds\\
&\ge \omega_{N-1}r^{N-\beta}\int_0^{(\log n)^{1-1/N}\|\widetilde{M}_n\|_E^{-1}}(\log n)^{1/N}\|\widetilde{M}_n\|_E\\
&\qquad e^{-(N-\beta)\|\widetilde{M}_n\|_E(\log n)^{1/N}s}ds\\
&= \frac{\omega_{N-1}r^{N-\beta}}{N-\beta}\left(1-e^{-(N-\beta)\log n}\right).
\end{aligned}
$$

Here we have used the change of variable $t=re^{-\|\widetilde{M}_n\|_E(\log n)^{1/N}s}$ in the third equality. Hence we obtain by passing to the limit $n\to\infty$ in (3.17)

$$
\liminf_{n\to\infty}\int_{|x|\le r}\frac{e^{\alpha_0|t_nM_n|^{\frac{N}{N-1}}}}{|x|^\beta}dx \ge \frac{\omega_{N-1}r^{N-\beta}}{N-\beta}\left(1+e^{-(N-\beta)m(r)\frac{(N-2)!}{N^N}r^N}\right).
$$

This together with (3.13)-(3.16) implies

$$
\left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}\ge(\beta_0-\eta)\frac{\omega_{N-1}r^{N-\beta}}{N-\beta}e^{-(N-\beta)m(r)\frac{(N-2)!}{N^N}r^N}.
$$

Since $\eta>0$ is arbitrary, we have

$$
\beta_0\le\frac{(N-\beta)^N}{\alpha_0^{N-1}r^{N-\beta}}e^{(N-\beta)m(r)\frac{(N-2)!}{N^N}r^N}.
$$

This contradicts (3.6) and ends the proof of (3.4).

Secondly it follows from (3.4) and the definition of $J_{\beta,\epsilon}$ that (3.5) holds. □

In this subsection, we will show that the weak limit of a Palais-Smale sequence for $J_{\beta,\epsilon}$ is the weak solution of (1.10). (Respectively the weak limit of a Palais-Smale sequence for J is also the weak solution of (1.3).)

Lemma 3.4. *Assume that* (V_1), (V_2), (H_1), (H_2) *and* (H_3) *are satisfied. Let* $(u_n) \subset E$ *be an arbitrary Palais-Smale sequence of* $J_{\beta,\epsilon}$, *i.e.,*

$$J_{\beta,\epsilon}(u_n) \to c, \ \ J'_{\beta,\epsilon}(u_n) \to 0 \text{ in } E^* \text{ as } n \to \infty, \tag{3.18}$$

where E^* *denotes the dual space of* E. *Then there exist a subsequence of* (u_n) *(still denoted by* (u_n)*) and* $u \in E$ *such that* $u_n \rightharpoonup u$ *weakly in* E, $u_n \to u$ *strongly in* $L^q(\mathbb{R}^N)$ *for all* $q \geq 1$, *and*

$$\begin{cases} \nabla u_n(x) \to \nabla u(x) \quad \text{a. e. in} \quad \mathbb{R}^N \\ \frac{f(x,u_n)}{|x|^\beta} \to \frac{f(x,u)}{|x|^\beta} \text{ strongly in } L^1(\mathbb{R}^N) \\ \frac{F(x,u_n)}{|x|^\beta} \to \frac{F(x,u)}{|x|^\beta} \text{ strongly in } L^1(\mathbb{R}^N). \end{cases}$$

Furthermore u *is a weak solution of (1.10). The same conclusion holds when* $\epsilon = 0$.

Proof. Assume (u_n) is a Palais-Smale sequence of $J_{\beta,\epsilon}$. By (3.18), we have

$$\frac{1}{N}\|u_n\|_E^N - \int_{\mathbb{R}^N} \frac{F(x,u_n)}{|x|^\beta}dx - \epsilon \int_{\mathbb{R}^N} hu_n dx \to c \text{ as } n \to \infty, \tag{3.19}$$

$$\left|\int_{\mathbb{R}^N} \left(|\nabla u_n|^{N-2}\nabla u_n \nabla\psi + V|u_n|^{N-2}u_n\psi\right)dx - \int_{\mathbb{R}^N} \frac{f(x,u_n)}{|x|^\beta}\psi dx - \epsilon \int_{\mathbb{R}^N} h\psi dx\right| \leq \tau_n \|\psi\|_E \tag{3.20}$$

for all $\psi \in E$, where $\tau_n \to 0$ as $n \to \infty$. Noticing that (1.12), we have by (H_2) that $0 \leq \mu F(x,u_n) \leq u_n f(x,u_n)$ for some $\mu > N$. Taking $\psi = u_n$ in (3.20) and multiplying (3.19) by μ, we have

$$\begin{aligned} \left(\frac{\mu}{N} - 1\right)\|u_n\|_E^N &\leq \left(\frac{\mu}{N} - 1\right)\|u_n\|_E^N - \int_{\mathbb{R}^N} \frac{\mu F(x,u_n) - f(x,u_n)u_n}{|x|^\beta}dx \\ &\leq \mu|c| + \tau_n\|u_n\|_E + (\mu+1)\epsilon\|h\|_{E^*}\|u_n\|_E \end{aligned}$$

Therefore $\|u_n\|_E$ is bounded. It then follows from (3.19), (3.20) that

$$\int_{\mathbb{R}^N} \frac{f(x,u_n)u_n}{|x|^\beta}dx \leq C, \quad \int_{\mathbb{R}^N} \frac{F(x,u_n)}{|x|^\beta}dx \leq C \tag{3.21}$$

for some constant C depending only on μ, N and $\|h\|_{E^*}$. By Lemma 2.4, up to a subsequence, $u_n \to u$ strongly in $L^q(\mathbb{R}^N)$ for some $u \in E$, $\forall q \geq 1$. This immediately leads to $u_n \to u$ almost everywhere in $\mathbb{R}^N$. Now we *claim* that up to a subsequence

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{|f(x,\, u_n) - f(x,\, u)|}{|x|^\beta}dx = 0. \tag{3.22}$$

In fact, since $f(x,\cdot) \geq 0$, it suffices to prove that up to a subsequence

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{f(x,\, u_n)}{|x|^\beta}dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{f(x,\, u)}{|x|^\beta}dx. \tag{3.23}$$

Since u, $\frac{f(x,u)}{|x|^\beta} \in L^1(\mathbb{R}^N)$, we have

$$\lim_{\eta \to +\infty} \int_{|u| \geq \eta} \frac{f(x,u)}{|x|^\beta} dx = 0.$$

Let C be the constant in (3.21). Given any $\delta > 0$, one can select some $M > C/\delta$ such that

$$\int_{|u| \geq M} \frac{f(x,u)}{|x|^\beta} dx < \delta. \tag{3.24}$$

It follows from (3.21) that

$$\int_{|u_n| \geq M} \frac{f(x,u_n)}{|x|^\beta} dx \leq \frac{1}{M} \int_{|u_n| \geq M} \frac{f(x,u_n)u_n}{|x|^\beta} dx < \delta. \tag{3.25}$$

For all $x \in \{x \in \mathbb{R}^N : |u_n| < M\}$, by our assumption (H_1), there exists a constant C_1 depending only on M such that $|f(x,u_n(x))| \leq C_1 |u_n(x)|^{N-1}$. Notice that $|x|^{-\beta}|u_n|^{N-1} \to |x|^{-\beta}|u|^{N-1}$ strongly in $L^1(\mathbb{R}^N)$ and $u_n \to u$ almost everywhere in $\mathbb{R}^N$. By the generalized Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n \to \infty} \int_{|u_n| < M} \frac{f(x,u_n)}{|x|^\beta} dx = \int_{|u| < M} \frac{f(x,u)}{|x|^\beta} dx. \tag{3.26}$$

Combining (3.24), (3.25) and (3.26), we can find some $K > 0$ such that when $n > K$,

$$\left| \int_{\mathbb{R}^N} \frac{f(x,u_n)}{|x|^\beta} dx - \int_{\mathbb{R}^N} \frac{f(x,u)}{|x|^\beta} dx \right| < 3\delta.$$

Hence (3.23) holds and thus our claim (3.22) holds. By (H_1) and (H_3), there exist constants c_1, $c_2 > 0$ such that

$$F(x,u_n) \leq c_1 |u_n|^N + c_2 f(x,u_n).$$

In view of (3.22) and Lemma 2.4, it follows from the generalized Lebesgue's dominated convergence theorem

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|F(x,\ u_n) - F(x,\ u)|}{|x|^\beta} dx = 0.$$

Using the argument of proving (4.26) in [5], we have $\nabla u_n(x) \to \nabla u(x)$ a. e. in $\mathbb{R}^N$ and

$$|\nabla u_n|^{N-2}\nabla u_n \rightharpoonup |\nabla u|^{N-2}\nabla u \quad \text{weakly in} \quad \left(L^{\frac{N}{N-1}}(\mathbb{R}^N)\right)^N.$$

Finally passing to the limit $n \to \infty$ in (3.20), we have

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{N-2} \nabla u \nabla \psi + V|u|^{N-2} u \psi \right) dx - \int_{\mathbb{R}^N} \frac{f(x,u)}{|x|^\beta} \psi dx - \epsilon \int_{\mathbb{R}^4} h \psi dx = 0$$

for all $\psi \in C_0^\infty(\mathbb{R}^N)$, which is dense in E. Hence u is a weak solution of (1.10). After checking the above argument, ϵ need not to be nonzero, i.e. the same conclusion holds for J. $\square$

Remark 3.5. Similar results of Lemma 3.4 was also established by J. M. do Ó in two dimensional case [20] and by the author for bi-Laplace equation in four dimensional Euclidean space [36].

4. Nontrivial positive solution

In this section, we will prove Theorem 1.1. It suffices to look for nontrivial critical points of the functional J in the function space E.

Proof of Theorem 1.1. By (i) and (ii) of Lemma 3.1, J satisfies all the hypothesis of the mountain-pass theorem except for the Palais-Smale condition: $J \in C^1(E, \mathbb{R})$; $J(0) = 0$; $J(u) \geq \delta > 0$ when $\|u\|_E = r$; $J(e) < 0$ for some $e \in E$ with $\|e\|_E > r$. Then using the mountain-pass theorem without the Palais-Smale condition [32], we can find a sequence (u_n) in E such that

$$J(u_n) \to c > 0, \quad J'(u_n) \to 0 \text{ in } E^*,$$

where

$$c = \min_{\gamma \in \Gamma} \max_{u \in \gamma} J(u) \geq \delta$$

is the min-max value of J, where $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}$. By (3.1), this is equivalent to saying

$$\frac{1}{N}\|u_n\|_E^N - \int_{\mathbb{R}^N} \frac{F(x, u_n)}{|x|^\beta} dx \to c \text{ as } n \to \infty, \tag{4.1}$$

$$\left| \int_{\mathbb{R}^N} \left(|\nabla u_n|^{N-2} \nabla u_n \nabla \psi + V |u_n|^{N-2} u_n \psi \right) dx - \int_{\mathbb{R}^N} \frac{f(x, u_n)}{|x|^\beta} \psi dx \right| \leq \tau_n \|\psi\|_E \tag{4.2}$$

for all $\psi \in E$, where $\tau_n \to 0$ as $n \to \infty$. By Lemma 3.4, up to a subsequence, there holds

$$\begin{cases} u_n \rightharpoonup u \text{ weakly in } E \\ u_n \to u \text{ strongly in } L^q(\mathbb{R}^N), \ \forall q \geq 1 \\ \lim\limits_{n \to \infty} \int_{\mathbb{R}^N} \frac{F(x,u_n)}{|x|^\beta} dx = \int_{\mathbb{R}^N} \frac{F(x,u)}{|x|^\beta} dx \\ u \text{ is a weak solution of (1.3).} \end{cases} \tag{4.3}$$

Now suppose by contradiction $u \equiv 0$. Since $F(x, 0) = 0$ for all $x \in \mathbb{R}^N$, it follows from (4.1) and (4.3) that

$$\lim_{n \to \infty} \|u_n\|_E^N = Nc > 0. \tag{4.4}$$

Thanks to the hypothesis (H_5), we have $0 < c < \frac{1}{N} \left(\frac{N-\beta}{N} \frac{\alpha_N}{\alpha_0} \right)^{N-1}$ by applying Lemma 3.3. Thus there exists some $\eta_0 > 0$ and $K > 0$ such that $\|u_n\|_E^N \leq \left(\frac{N-\beta}{N} \frac{\alpha_N}{\alpha_0} - \eta_0 \right)^{N-1}$ for all $n > K$. Choose $q > 1$ sufficiently close to 1 such that $q\alpha_0 \|u_n\|_E^{\frac{N}{N-1}} \leq (1 - \beta/N)\alpha_N - \alpha_0\eta_0/2$ for all $n > N$. By (H_1),

$$|f(x, u_n) u_n| \leq b_1 |u_n|^N + b_2 |u_n| \zeta \left(N, \alpha_0 |u_n|^{\frac{N}{N-1}} \right),$$

where the function $\zeta(\cdot, \cdot)$ is defined by (2.1). It follows from the Hölder inequality, Lemma 2.1

and Theorem A that

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{|f(x,u_n)u_n|}{|x|^\beta}dx &\le b_1\int_{\mathbb{R}^N}\frac{|u_n|^N}{|x|^\beta}dx + b_2\int_{\mathbb{R}^N}\frac{|u_n|\zeta\left(N,\alpha_0|u_n|^{\frac{N}{N-1}}\right)}{|x|^\beta}dx\\
&\le b_1\int_{\mathbb{R}^N}\frac{|u_n|^N}{|x|^\beta}dx + b_2\left(\int_{\mathbb{R}^N}\frac{|u_n|^{q'}}{|x|^\beta}dx\right)^{1/q'}\left(\int_{\mathbb{R}^N}\frac{\zeta\left(N,q\alpha_0|u_n|^{\frac{N}{N-1}}\right)}{|x|^\beta}dx\right)^{1/q}\\
&\le b_1\int_{\mathbb{R}^N}\frac{|u_n|^N}{|x|^\beta}dx + C\left(\int_{\mathbb{R}^N}\frac{|u_n|^{q'}}{|x|^\beta}dx\right)^{1/q'} \to 0 \quad \text{as} \quad n\to\infty.
\end{aligned}$$

Here we used (4.3) again (precisely $u_n \to u$ in $L^s(\mathbb{R}^N)$ for all $s \ge 1$) in the last step of the above estimates. Inserting this into (4.2) with $\psi = u_n$, we have

$$\|u_n\|_E \to 0 \quad \text{as} \quad n \to \infty,$$

which contradicts (4.4). Therefore $u \not\equiv 0$ and we obtain a nontrivial weak solution of (1.3). □

5. Multiplicity results

In this section we will prove Theorem 1.2. The proof is divided into three steps, namely

Step 1. *Let ϵ_1 be given by (ii) of Lemma 3.1, and ϵ^*, δ^* be given by Lemma 3.3. Then when $0 < \epsilon < \epsilon_1$, there exists a sequence $(v_n) \subset E$ such that*

$$J_{\beta,\epsilon}(v_n) \to c_M, \quad J'_{\beta,\epsilon}(v_n) \to 0, \tag{5.1}$$

where c_M is a min-max value of $J_{\beta,\epsilon}$. Let $\epsilon_2 = \min\{\epsilon_1, \epsilon^*\}$. Then when $0 < \epsilon < \epsilon_2$, we can take c_M such that

$$0 < c_M < \frac{1}{N}\left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1} - \delta^*. \tag{5.2}$$

In addition, up to a subsequence, there holds $v_n \rightharpoonup u_M$ weakly in E, and u_M is a weak solution of (1.10).

Proof. By (*i*) and (*ii*) of Lemma 3.1, when $0 < \epsilon < \epsilon_1$, $J_{\beta,\epsilon}$ satisfies the following condition: $J_{\beta,\epsilon} \in C^1(E,\mathbb{R})$; $J_{\beta,\epsilon}(0) = 0$; $J_{\beta,\epsilon}(u) \ge \vartheta_\epsilon > 0$ when $\|u\|_E = r_\epsilon$; $J_{\beta,\epsilon}(e) < 0$ for some $e \in E$ with $\|e\| > \max\{r_\epsilon, 1\}$. Then using the mountain-pass theorem without the Palais-Smale condition [32], we can find a sequence (v_n) in E such that

$$J_{\beta,\epsilon}(v_n) \to c_M > 0, \quad J'_{\beta,\epsilon}(v_n) \to 0 \text{ in } E^*,$$

where

$$c_M = \min_{\gamma\in\Gamma}\max_{u\in\gamma} J_{\beta,\epsilon}(u) \ge \vartheta_\epsilon$$

is a min-max value of $J_{\beta,\epsilon}$, where $\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0, \gamma(1) = e\}$. Clearly (5.2) follows from Lemma 3.3. The last assertion follows from Lemma 3.4 immediately. □

Step 2. *Let r_ϵ be given by (ii) of Lemma 3.1 such that $r_\epsilon \to 0$ as $\epsilon \to 0$. There exists $\epsilon_3 : 0 < \epsilon_3 < \epsilon_2$ such that if $0 < \epsilon < \epsilon_3$, then there exists a sequence $(u_n) \subset E$ such that*

$$J_{\beta,\epsilon}(u_n) \to c_\epsilon := \inf_{\|u\|_E \le r_\epsilon} J_{\beta,\epsilon}(u) \tag{5.3}$$

and

$$J'_{\beta,\epsilon}(u_n) \to 0 \quad \text{in} \quad E^* \quad \text{as} \quad n \to \infty, \tag{5.4}$$

where $c_\epsilon < 0$ and $c_\epsilon \to 0$ as $\epsilon \to 0$. In addition, up to a subsequence, there holds $u_n \to u_0$ strongly in E, and u_0 is a weak solution of (1.10) with $J_{\beta,\epsilon}(u_0) = c_\epsilon$.

Proof. Let r_ϵ be given by (ii) of Lemma 3.1, i.e. $J_{\beta,\epsilon}(u) > \vartheta_\epsilon > 0$ for all u with $\|u\|_E = r_\epsilon$. Since $r_\epsilon \to 0$ as $\epsilon \to 0$, one can choose $\epsilon_3 : 0 < \epsilon_3 < \epsilon_2$ such that when $0 < \epsilon < \epsilon_3$,

$$r_\epsilon < \left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{\frac{N-1}{N}}. \tag{5.5}$$

By (H_1) and (H_2), we have

$$F(x,u) \le b_1|u|^N + b_2|u|\zeta\left(N, \alpha_0\|u\|_E^{N/(N-1)}\left(|u|/\|u\|_E\right)^{N/(N-1)}\right). \tag{5.6}$$

Here again $\zeta(\cdot,\cdot)$ is defined by (2.1). When $\|u\|_E \le r_\epsilon$, we have $\alpha_0\|u\|_E^{N/(N-1)} < (1-\beta/N)\alpha_N$. It then follows from Lemma 2.1 and Theorem A that $F(x,u)/|x|^\beta$ is bounded in $L^p(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ for some $p > 1$ when $\|u\|_E \le r_\epsilon$. Hence $J_{\beta,\epsilon}$ has lower bound on the ball $B_{r_\epsilon} = \{u \in E : \|u\|_E \le r_\epsilon\}$.

Since the closure of B_{r_ϵ}, $\overline{B}_{r_\epsilon} \subset E$ is a complete metric space with the metric given by the norm of E, convex and $J_{\beta,\epsilon}$ is of class C^1 and has lower bound on $\overline{B}_{r_\epsilon}$. By the Ekeland's variational principle [35], there exists a sequence $(u_n) \subset \overline{B}_{r_\epsilon}$ such that (5.3) and (5.4) hold.

By (iii) of Lemma 3.1, $c_\epsilon < 0$. Since $r_\epsilon \to 0$ as $\epsilon \to 0$, noticing (5.6), we have by using the Hölder inequality and Lemma 2.4

$$\sup_{\|u\|_E \le r_\epsilon} \int_{\mathbb{R}^N} \frac{F(x,u)}{|x|^\beta} dx \to 0, \quad \sup_{\|u\|_E \le r_\epsilon} \int_{\mathbb{R}^N} hu dx \to 0$$

as $\epsilon \to 0$. This implies $c_\epsilon \to 0$ as $\epsilon \to 0$.

Now we are proving the last assertion. Assume $u_n \rightharpoonup u_0$ weakly in E. (5.4) is equivalent to

$$|\langle J'_{\beta,\epsilon}(u_n), \phi\rangle| \le \tau_n\|\phi\|_E, \quad \forall \phi \in E, \tag{5.7}$$

where $\tau_n \to 0$ as $n \to \infty$. Recalling (3.2) and choosing $\phi = u_n - u_0$ in (5.7), we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(|\nabla u_n|^{N-2}\nabla u_n \nabla(u_n - u_0) + V(x)|u_n|^{N-2}u_n(u_n - u_0)\right) dx \\ &- \int_{\mathbb{R}^N} \frac{f(x,u_n)}{|x|^\beta}(u_n - u_0)dx - \epsilon \int_{\mathbb{R}^N} h(u_n - u_0)dx = o_n(1), \end{aligned}$$

where $o_n(1) \to 0$ as $n \to \infty$. Hölder inequality together with (5.5), Theorem A and Lemma 2.4 implies that

$$\int_{\mathbb{R}^N} \frac{f(x,u_n)}{|x|^\beta}(u_n - u_0)dx = o_n(1), \quad \epsilon \int_{\mathbb{R}^N} h(u_n - u_0)dx = o_n(1).$$

Hence
$$\int_{\mathbb{R}^N} \left(|\nabla u_n|^{N-2}\nabla u_n \nabla(u_n-u_0) + V(x)|u_n|^{N-2}u_n(u_n-u_0)\right)dx = o_n(1). \tag{5.8}$$
On the other hand, since $u_n \rightharpoonup u_0$ weakly in E, we obtain
$$\int_{\mathbb{R}^N} \left(|\nabla u_0|^{N-2}\nabla u_0 \nabla(u_n-u_0) + V(x)|u_0|^{N-2}u_0(u_n-u_0)\right)dx = o_n(1). \tag{5.9}$$
Subtracting (5.9) from (5.8), using a well known inequality (see for example Chapter 10 of [27])
$$2^{2-N}|b-a|^N \le \langle |b|^{N-2}b - |a|^{N-2}a, b-a\rangle, \quad \forall a,\, b \in \mathbb{R}^N, \tag{5.10}$$
we obtain $\|u_n-u_0\|_E^N \to 0$ and thus $u_n \to u_0$ strongly in E as $n\to\infty$. Since $J_{\beta,\epsilon} \in C^1(E,\mathbb{R})$, there hold $J_{\beta,\epsilon}(u_0) = c_\epsilon$ and $J'_{\beta,\epsilon}(u_0) = 0$, i.e. u_0 is a weak solution of (1.10). □

Step 3. *There exists* $\epsilon_0 : 0 < \epsilon_0 < \epsilon_3$ *such that if* $0 < \epsilon < \epsilon_0$, *then* $u_M \not\equiv u_0$.

Proof. Suppose by contradiction that $u_M \equiv u_0$. Then $v_n \rightharpoonup u_0$ weakly in E. By (5.1),
$$J_{\beta,\epsilon}(v_n) \to c_M > 0, \quad |\langle J'_{\beta,\epsilon}(v_n),\phi\rangle| \le \gamma_n\|\phi\|_E \tag{5.11}$$
with $\gamma_n \to 0$ as $n\to\infty$. On one hand, by Lemma 3.4, we have
$$\int_{\mathbb{R}^N} \frac{F(x,v_n)}{|x|^\beta}dx \to \int_{\mathbb{R}^N} \frac{F(x,u_0)}{|x|^\beta}dx \quad \text{as} \quad n\to\infty. \tag{5.12}$$
Here and in the sequel, we do not distinguish sequence and subsequence. On the other hand, since $v_n \rightharpoonup u_0$ weakly in E, it follows from the Hölder inequality and Lemma 2.4 that
$$\int_{\mathbb{R}^N} hv_n dx \to \int_{\mathbb{R}^N} hu_0 dx. \quad \text{as} \quad n\to\infty. \tag{5.13}$$
Inserting (5.12) and (5.13) into (5.11), we obtain
$$\frac{1}{N}\|v_n\|_E^N = c_M + \int_{\mathbb{R}^N} \frac{F(x,u_0)}{|x|^\beta}dx + \epsilon\int_{\mathbb{R}^N} hu_0dx + o_n(1). \tag{5.14}$$
In the same way, one can derive
$$\frac{1}{N}\|u_n\|_E^N = c_\epsilon + \int_{\mathbb{R}^N} \frac{F(x,u_0)}{|x|^\beta}dx + \epsilon\int_{\mathbb{R}^N} hu_0dx + o_n(1). \tag{5.15}$$
Combining (5.14) and (5.15), we have
$$\|v_n\|_E^N - \|u_0\|_E^N = N\left(c_M - c_\epsilon + o_n(1)\right). \tag{5.16}$$
From Step 2, we know that $c_\epsilon \to 0$ as $\epsilon \to 0$. This together with (5.2) leads to the existence of $\epsilon_0 : 0 < \epsilon_0 < \epsilon_3$ such that if $0 < \epsilon < \epsilon_0$, then
$$0 < c_M - c_\epsilon < \frac{1}{N}\left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}. \tag{5.17}$$

Write

$$w_n = \frac{v_n}{\|v_n\|_E}, \quad w_0 = \frac{u_0}{\left(\|u_0\|_E^N + N(c_M - c_\epsilon)\right)^{1/N}}.$$

It follows from (5.16) and $v_n \rightharpoonup u_0$ weakly in E that $w_n \rightharpoonup w_0$ weakly in E. Notice that

$$\int_{\mathbb{R}^N} \frac{\zeta\left(N, \alpha_0 |v_n|^{N/(N-1)}\right)}{|x|^\beta} dx = \int_{\mathbb{R}^N} \frac{\zeta\left(N, \alpha_0 \|v_n\|_E^{N/(N-1)} |w_n|^{N/(N-1)}\right)}{|x|^\beta} dx.$$

By (5.16) and (5.17), a straightforward calculation shows

$$\lim_{n\to\infty} \alpha_0 \|v_n\|_E^{\frac{N}{N-1}} \left(1 - \|w_0\|_E^N\right)^{\frac{1}{N-1}} < \left(1 - \frac{\beta}{N}\right) \alpha_N.$$

Whence Lemma 2.3 together with Lemma 2.1 implies that $\zeta\left(N, \alpha_0 |v_n|^{N/(N-1)}\right)/|x|^\beta$ is bounded in $L^q(\mathbb{R}^N)$ for some $q > 1$. By (H_1),

$$|f(x, v_n)| \leq b_1 |v_n|^{N-1} + b_2 \zeta(N, \alpha_0 |v_n|^{\frac{N}{N-1}}).$$

Then it follows from the continuous embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ for all $p \geq 1$ that $f(x, v_n)/|x|^\beta$ is bounded in $L^{q_1}(\mathbb{R}^N)$ for some q_1: $1 < q_1 < q$. This together with Lemma 2.4 and the Hölder inequality gives

$$\left| \int_{\mathbb{R}^N} \frac{f(x, v_n)(v_n - u_0)}{|x|^\beta} dx \right| \leq \left\| \frac{f(x, v_n)}{|x|^\beta} \right\|_{L^{q_1}(\mathbb{R}^N)} \|v_n - u_0\|_{L^{q_1'}(\mathbb{R}^N)} \to 0, \tag{5.18}$$

where $1/q_1 + 1/q_1' = 1$.

Taking $\phi = v_n - u_0$ in (5.11), we have by using (5.13) and (5.18) that

$$\int_{\mathbb{R}^N} \left(|\nabla v_n|^{N-2} \nabla v_n \nabla (v_n - u_0) + V(x) |v_n|^{N-2} v_n (v_n - u_0)\right) dx \to 0. \tag{5.19}$$

However the fact $v_n \rightharpoonup u_0$ weakly in E leads to

$$\int_{\mathbb{R}^N} \left(|\nabla u_0|^{N-2} \nabla u_0 \nabla (v_n - u_0) + V(x) |u_0|^{N-2} u_0 (v_n - u_0)\right) dx \to 0. \tag{5.20}$$

Subtracting (5.20) from (5.19), using the inequality (5.10), we have

$$\|v_n - u_0\|_E^N \to 0.$$

This together with (5.16) implies that

$$c_M = c_\epsilon,$$

which is absurd since $c_M > 0$ and $c_\epsilon < 0$. Therefore we end Step 3 and complete the proof of Theorem 1.2. □

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